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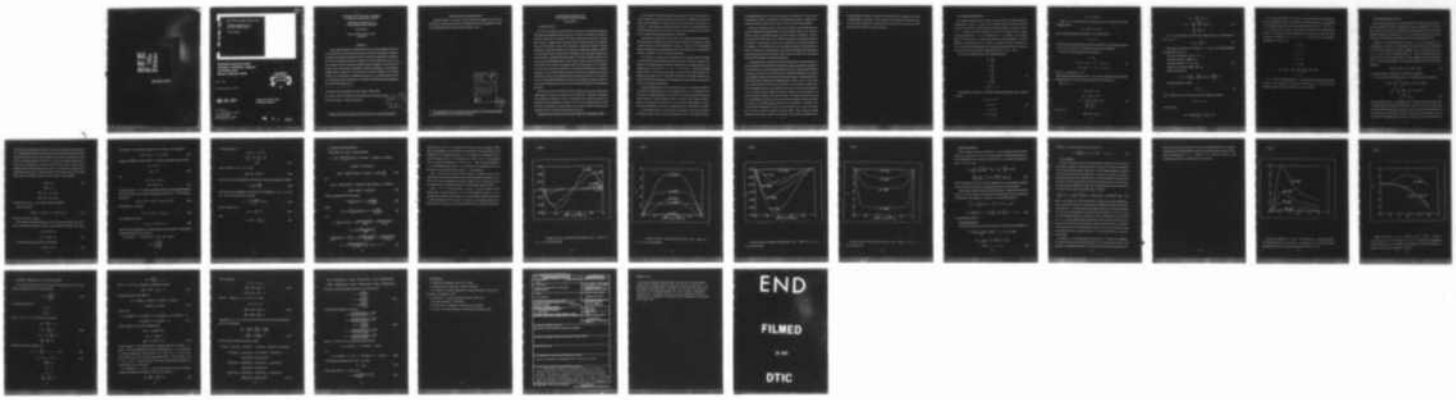

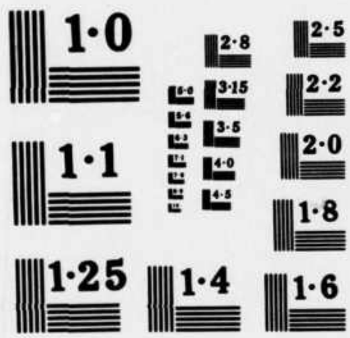

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INTERFACIAL STABILITY IN A TWO-LAYER BÉNARD PROBLEM

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ABSTRACT

A linear stability analysis of the Bénard problem for two layers of different fluids lying on top of each other and bounded by free surfaces is considered. The fluids are assumed to be similar and perturbation methods are used to calculate the eigenvalue in closed form. The case of the Rayleigh number and wavenumber of the disturbance being close to the first criticality of the one-fluid Bénard problem has been investigated in a previous paper[1], and was found to exhibit both overstability and convective instability. In this paper, the Rayleigh number is assumed to be less than that of the first criticality of the one-fluid problem, and in this situation, overstability does not occur. An unexpected result is that by an appropriate choice of parameters, it is possible to find linearly stable arrangements with the more dense fluid on top.

AMS (MOS) Subject Classifications: 76E15, 76E20, 76T05, 76V05

Key Words: Convective instability; Two-component flow; Interfacial stability; Navier Stokes equations; density; Boussinesq approximation,

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SIGNIFICANCE AND EXPLANATION

Interfacial stability of a two-layer convective problem is analyzed for the case of two similar liquids. Applications may involve the modeling of convective instability in a variety of multi-layered films, for example, pattern formation in rocks.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

INTERFACIAL STABILITY IN A TWO-LAYER BÉNARD PROBLEM

Yuriko Renardy

I. INTRODUCTION

Two layers of fluids are sandwiched between infinite parallel horizontal boundaries. The fluids have only slightly different properties. We neglect any diffusion of one fluid into the other, so that there is an interface, with surface tension taken into account. The lower plate is kept at a slightly higher temperature than the upper plate, and the tangential stress and normal velocity vanish at the boundaries. Although the "free-free" boundary conditions are physically unrealistic, they yield the advantage that closed-form solutions can be obtained, and their analysis helps to shed light into the numerical investigation of the more realistic "rigid-rigid" boundary problem[2]. We consider the linear stability of the rest state when the Rayleigh number is below the first criticality of the corresponding one-fluid problem. The case when the Rayleigh number and wavenumber of the disturbance are close to that of the first criticality of the one-fluid problem has been investigated previously[1]. For the linear stability analysis, we only need to consider two-dimensional disturbances, because of the rotational symmetry about the vertical axis. The questions we ask are: what kinds of instabilities can arise; are they different or similar to those that arise in the one-fluid problem; and how do these instabilities depend on the dimensionless parameters.

The equations used in each fluid are the Navier-Stokes equations with the Boussinesq approximation, the linear heat equation and incompressibility. Temperature differences in the problem are assumed to be small. The properties of the fluids are assumed to be constants except for the density in the gravity term in the Navier-Stokes equations, where the density is expanded as a Taylor series about the temperature of the top boundary, and truncated so that it is a linear function of the temperature.

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At the interface, the following conditions are assumed to hold: the kinematic free-surface condition, the continuity of velocity and shear stress, the difference in the normal stresses must be balanced by surface tension, and the temperature and heat flux are continuous. These equations yield 9 dimensionless parameters: a Rayleigh number and Prandtl number based on one of the fluids, a surface tension parameter, and 6 ratios of the various thermal and mechanical properties of the fluids.

One solution to the equations is the rest state, with a linear temperature gradient in each fluid, and a flat interface. We consider the linear stability of this solution by adding a small perturbation which is proportional to $\exp(i\alpha x + \sigma t)$ where x is the dimensionless horizontal variable and t is the dimensionless time. The problem is set up as an eigenvalue problem, to calculate σ in terms of the other parameters.

It is instructive to recall some of the results for the one-fluid problem. Rayleigh, in 1916,[3] solved the linear stability problem for one fluid, for "free-free" boundaries. The eigenvalues and eigenfunctions were found in closed form, and the marginal stability curves are given by $R_j = (j^2\pi^2+\alpha^2)^3/\alpha^2$, $j = 1,2,\ldots$, $R_1 < R_2 < R_3 < \ldots$ The critical Rayleigh number is $27\pi^4/4$. The Prandtl number does not enter into the criticality condition, but appears in the growth rate. The "rigid-rigid" boundary and "free-rigid" boundary cases were solved later[3]: here, the eigenvalues and eigenfunctions cannot be found in closed form but have to be calculated numerically.

Another well-known result for the one-fluid problem is the "exchange of stabilities": the equations are self-adjoint so that all the eigenvalues are real. A consequence is that the velocity at marginal instability is not periodic. The exponential growth of small disturbances leads to a steady nonlinear motion, for example, steady cellular convection. However, in the presence of solutes with one or more concentration gradients, or continuous density-stratification, a time-periodic marginal instability ("overstability") is possible.

In the two-fluid problem, the equations are not self-adjoint[2]. Therefore, overstability is possible, and such a situation was found numerically for the case of rigid boundaries.

The marginal eigenvalues are a complex conjugate pair of multiplicity two. The rest state would bifurcate to either a traveling wave or a standing wave solution. An analysis of the subsequent nonlinear problem may be related to that[4] of one-fluid double diffusion.

It would be interesting to find out under what conditions complex eigenvalues arise and what conditions yield real eigenvalues in the linear stability problem, but a numerical attempt at this is not feasible because of the large number of parameters. However, the case of free boundaries where the two fluids have thermal and mechanical properties that differ by a small amount of $O(\epsilon)$ can be analyzed with perturbation methods in the parameter ϵ. The leading terms in the perturbation expansion for the eigenvalue are found in closed form. In a previous paper[1], this analysis was performed for the situation where the unperturbed one-fluid problem is close to the first criticality. Thus, the analysis involved the perturbation of a double zero eigenvalue.

In the present paper, the unperturbed one-fluid problem is below the first criticality, so that the analysis for two similar fluids involves the perturbation of a simple zero eigenvalue. Since the equations are real, the perturbed eigenvalue is also real. Hence, when the interface is unstable, one would expect the fluids to go into a different arrangement. For example, if the instability results from the upper fluid being the heavier, one would expect that it should fall and the fluids should exchange positions. However, there are more subtle cases as will be discussed. The short-wave asymptotics for σ is identical to that obtained for the rigid-rigid boundaries[2], and indeed this agrees with the short-wave asymptotics of our perturbation formula. Differences in density, coefficients of cubical expansion and surface tension are important in the short-wave limit. In the long-wave limit, the volume ratio, and differences in thermal conductivity, density and coefficients of cubical expansion are important. By making appropriate choices for the parameters, it is possible to find a linearly stable arrangement with the more dense fluid on top. In fact, the formula for σ for small l_1 shows the unexpected "thin-layer effect", where stability is strongly influenced by the thermal conductivity stratification. This behavior is reminiscent of the

lubricating effect of viscosity stratification in parallel shear flows composed of two iayers of different fluids[5], where linearly stable arrangements with the heavier fluid lying on top have been found, provided that suitable values are chosen for the volume ratio, surface tension. viscosity difference and density difference.

II. GOVERNING EQUATIONS

We consider two fluids, lying on top of each other between two parallel boundaries of infinite extent in the (x^*, z^*)-plane. Fluid 1 denotes the lower of the fluids and Fluid 2 denotes the upper fluid. Asterisks denote dimensional variables. The upper boundary at $z^* = l^*$ is kept at a constant temperature T_0^*, and the lower plate is kept at a higher constant temperature $T_0^* + \Delta T^*$. The interface is at $z = l_1^*$. Subscript i (=1,2) denotes Fluid i. At temperature T_0^*, the fluids have coefficients of cubical expansion $\hat{\alpha}_i$, thermal diffusivity κ_i, thermal conductivity k_i, viscosity μ_i, kinematic viscosity ν_i and density ρ_i. We use the Boussinesq approximation in the Navier-Stokes equations, that is, the densities in the buoyancy term for each fluid is approximated by $\rho_i(1 - \hat{\alpha}_i(T^* - T_0^*))$.

There are 6 dimensionless ratios of the physical properties:

$$
\begin{aligned}
m &= \frac{\mu_1}{\mu_2}, \\
r &= \frac{\rho_1}{\rho_2}, \\
\gamma &= \frac{\kappa_1}{\kappa_2}, \\
\varsigma &= \frac{k_1}{k_2}, \\
\beta &= \frac{\hat{\alpha}_1}{\hat{\alpha}_2}, \\
l_1 &= \frac{l_1^*}{l^*} = 1 - l_2.
\end{aligned}
\tag{1}
$$

Following Drazin and Reid [3], we introduce dimensionless variables (without asterisks) as follows:

$$
\begin{aligned}
(x, z) &= (x^*, z^*)/l^*, \\
t &= \kappa_1 t^*/l^{*2}, \\
\underline{u} &= \underline{u}^* l^*/\kappa_1, \\
T &= T^*/\Delta T^*,
\end{aligned}
\tag{2}
$$

$$p = p^{*} l^{*2} / (\rho_1 \kappa_1^2).$$

Here, $\underline{u}^{*} = (u^{*}, w^{*})$ is the velocity, p^{*} the pressure and T^{*} the temperature. We define a Rayleigh number

$$R = g \hat{\alpha}_1 \Delta T^{*} l^{*3} / (\kappa_1 \nu_1),$$

where g denotes gravitational acceleration, and a Prandtl number

$$P = \nu_1 / \kappa_1.$$

We include a surface tension between the fluids, described by a dimensionless parameter $S = S^{*} l^{*} / (\kappa_1 \mu_1)$, where S^{*} is the dimensional surface tension coefficient.

We study the linear stability of the static solution

$$\underline{u} = \underline{0},$$

$$\begin{aligned} T &= T_0 + 1 - A_1 z \qquad \text{for} \qquad 0 \le z \le l_1, \\ &= T_0 + A_2(1 - z) \qquad \text{for} \qquad l_1 \le z \le 1, \end{aligned} \tag{3}$$

where $A_1 = \frac{1}{(l_1 + \varsigma l_2)}$ and $A_2 = \varsigma A_1$.

If disturbances are proportional to $\exp(\sigma t + i \alpha x)$, then the following linearized eigenvalue problem arises for the velocity $\underline{u}$ and perturbation Θ to the temperature.

For $0 \le z \le l_1$,

$$\begin{aligned} \sigma \Theta &= w A_1 + \triangle \Theta, \\ \sigma u &= -\partial p / \partial x + P \triangle u. \\ \sigma w &= -\partial p / \partial z + R P \Theta + P \triangle w, \\ \frac{\partial u}{\partial x} &+ \frac{\partial w}{\partial z} = 0. \end{aligned} \tag{4}$$

For $l_1 \le z \le 1$,

$$\sigma \Theta = w A_2 + \frac{1}{\gamma} \triangle \Theta.$$

$$\sigma u = -r\frac{\partial p}{\partial x} + \frac{r}{m} P \triangle u,$$

$$\sigma w = -r\frac{\partial p}{\partial z} + \frac{RP}{\beta}\Theta + \frac{r}{m} P \triangle w, \tag{5}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0.$$

At $z = 0, 1$, slip boundary conditions are assumed to apply, i.e., the shear stress is zero, or, $\frac{\partial u}{\partial z} = 0$:

$$\Theta = w = \frac{\partial u}{\partial z} = 0. \tag{6}$$

The interface is perturbed to the position $z = l_1 + h(x,t)$. The interface conditions linearized about z=l_1 are[2,6] :

continuity of temperature: $[\![\Theta\ ]\!] = h\ [\![A\ ]\!]$,

continuity of heat flux: $[\![k\frac{\partial \Theta}{\partial z}\ ]\!]=0$,

continuity of velocity: $[\![w\ ]\!] = [\![u\ ]\!] = 0$, (7)

continuity of shear stress: $[\![\mu(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x})\ ]\!]=0$,

balance of normal stress:

$$p_2 - p_1 + 2P(\frac{\partial w_1}{\partial z} - \frac{1}{m}\frac{\partial w_2}{\partial z}) + M_1 h - PS\frac{\partial^2 h}{\partial x^2} = 0,$$

kinematic free surface condition:

$$\sigma h = w_1. \tag{8}$$

Here, $[\![.\ ]\!]$denotes the jump of a quantity across the interface. for example,

$$[\![A]\!] = A_1 - A_2,$$

and we have set

$$M_1 = RP\{\frac{(1-\frac{1}{r})}{\hat{\alpha}_1 \triangle T} + l_2 A_2(\frac{1}{r\beta} - 1)\}.$$

We now proceed as follows. It is known[2,3] that when the fluids have identical prop-
rties, the eigenvalues are real, the presence of the interface introduces a zero eigenvalue t any Rayleigh number, and another zero eigenvalue occurs first at $R = \frac{27}{4}\pi^4$ for $\alpha = {}^{-1/2}\pi$. We let R be arbitrary but less than the value of the first criticality of the one-fluid roblem. We assume the fluids have slightly different properties, and introduce a small arameter ϵ. We regard $1-m, 1-r, 1-\gamma, 1-\varsigma, 1-\beta, M_1$ and S as small quantities roportional to ϵ; that is, we set

$$1 - m = \bar{m}\epsilon,$$

$$1 - r = \bar{r}\epsilon,$$

$$1 - \gamma = \bar{\gamma}\epsilon,$$

$$1 - \varsigma = \bar{\varsigma}\epsilon,$$

$$1 - \beta = \bar{\beta}\epsilon,$$

$$M_1 = \bar{M}_1\epsilon, \quad \bar{M}_1 = PR\left(-\frac{\bar{r}}{\dot{\alpha}_1\Delta T^{\cdot}} + l_2(\bar{r} + \bar{\beta})\right),$$

$$S = \bar{S}\epsilon.$$

At $\epsilon = 0$, there is a simple eigenvalue $\sigma = 0$, arising from the presence of the interface. or small ϵ, this eigenvalue is perturbed into a power series in ϵ. The purpose of the llowing analysis is to find the coefficient of ϵ in this expansion.

III. PERTURBATION ANALYSIS

The perturbation expansion for a double eigenvalue is carried out in detail in Ref. 1, and closely parallels the procedure adopted here so that the explanation in this section will be terse. The perturbation expansion for the simple eigenvalue involves finding the eigenspace belonging to the eigenvalue $\sigma = 0$ for the *unperturbed* problem and its adjoint, and does *not* require finding the eigenspace of the perturbed $O(\epsilon)$ problem at all.

Suppose σ_0 is a simple eigenvalue of a matrix L_0. Let A be an eigenvector of L_0 with eigenvalue σ_0, and let C be an eigenvector of L_0^* (the adjoint of L_0) with eigenvalue $\bar{\sigma}_0$ (the overbar here denotes the complex conjugate). Let L_0 be perturbed into $L(\epsilon) = L_0 + \epsilon L_1 + O(\epsilon^2)$ with L_1 depending smoothly on ϵ. Then the perturbed eigenvalue σ is given by the zero of the expression $\Psi(\epsilon,\sigma)$, which represents to $O(\epsilon)$ the projection of $L(\epsilon) - \sigma$, first onto the eigenspace of the unperturbed problem and then onto the adjoint eigenspace:

$$\Psi(\epsilon,\sigma) = \langle C, (L_0 + \epsilon L_1 - \sigma)A\rangle + O(\epsilon^2). \tag{9}$$

For the same reasons as in §III, Ref. 1, we will need to redefine Ψ.

Let X denote the set of functions (Θ, u, w, h). We introduce an inner product by

$$\begin{aligned}\langle X_1, X_2\rangle = &\int_0^{2\pi/\alpha}\int_{z=0}^{l_1} \bar{\Theta}_1\Theta_2 + \bar{u}_1 u_2 + \bar{w}_1 w_2 \; dzdx \\ &+ \int_0^{2\pi/\alpha}\int_{z=l_1}^{1} \bar{\Theta}_1\Theta_2 + \bar{u}_1 u_2 + \bar{w}_1 w_2 \; dzdx \\ &\qquad + \int_0^{2\pi/\alpha} \bar{h}_1 h_2 \; dx\end{aligned} \tag{10}$$

to generate a Hilbert space. In this Hilbert space, we consider the subspace determined by the "Hodge projection" (see space H in Theorem 1.4, *Ref. 7*), that is, by the conditions that the velocity field be divergence-free, that the vertical velocity vanish at the boundaries, and be continuous across the interface. By $L(\epsilon)X$ we denote the right hand sides of equations (4),(5) and (8). We regard $L(\epsilon)$ as an operator in the subspace so that the conditions on w

in (6) and (7) and the normal stress balance in (7) are an integral part of the definition of L(ϵ). The domain of definition of L(ϵ) is determined by the rest of the boundary conditions in (6) and (7), which we write in the form B(ϵ)X=0. The range of the operator L(ϵ) must satisfy the following conditions in order for the pressure p occuring on the right sides of (4) and (5) to be determined as a function of X: The "velocity part" of L(ϵ)X must be divergence free, the vertical velocity must vanish on the walls and be continuous across the interface, and the jump in p across the interface must be given by the normal stress balance. Thus, the problem we wish to solve is: for small ϵ, find σ satisfying

$$L(\epsilon)X = \sigma X, \tag{11}$$

$$B(\epsilon)X = 0,$$

$$L(\epsilon) = L_0 + \epsilon L_1 + O(\epsilon^2),$$

$$B(\epsilon) = B_0 + \epsilon B_1 + O(\epsilon^2).$$

The explicit forms of L_0, L_1, B_0 and B_1 are given in §III, Ref. 1.

We now redefine Ψ^1:

$$\Psi(\epsilon,\hat{\sigma}) = \langle C,\ ((L(\epsilon) - 1)^{-1} - \hat{\sigma})A\rangle + O(\epsilon^2), \tag{12}$$

where the C and A are as before.

The boundary value problem adjoint to (11) is calculated in Appendix A of Ref. 1. Then we determine the eigenvectors A and C in the Appendix of this paper. They satisfy:

$$L_0 A = 0,\ B_0 A = 0,$$

$$L_0^{\cdot} C = 0,\ B_0^{\cdot} C = 0. \tag{13}$$

In formula (12), we must determine the expressions

$$\langle C, (L(\epsilon) - 1)^{-1} A\rangle \tag{14}$$

to first order in ϵ. To facilitate this calculation. we introduce x^0 and x^1 defined by

$$(L(\epsilon) - 1)^{-1} A = x^0 + \epsilon x^1 + O(\epsilon^2). \tag{15}$$

Equating the coefficients of equal powers of ϵ, we find the equations governing x^0 and x^1

$$\begin{aligned} (L_0 - 1)x^0 &= A. \\ B_0 x^0 &= 0, \end{aligned} \tag{16}$$

and

$$\begin{aligned} L_1 x^0 + (L_0 - 1)x^1 &= 0, \\ B_1 x^0 + B_0 x^1 &= 0. \end{aligned} \tag{17}$$

From (16), we find $x^0 = -A$. We will not need the solutions x^1 to the perturbation problem (17) but only certain inner products involving them, namely $\langle C, x^1 \rangle$. This is seen from (12) and (15):

$$\Psi(\epsilon, \hat{\sigma}) = \langle C, x^0 \rangle + \epsilon \langle C, x^1 \rangle - \hat{\sigma} \langle C, A \rangle + O(\epsilon^2). \tag{18}$$

We calculate $\langle C, x^1 \rangle$ from (17):

$$\langle C, x^1 \rangle = \langle C, L_1 x^0 \rangle + \langle C, L_0 x^1 \rangle, \tag{19}$$

and an integration by parts:

$$\langle C, L_0 x^1 \rangle = \langle L_0^* C, x^1 \rangle + \Gamma, \tag{20}$$

where the boundary integrals Γ are evaluated using the second part of (17). (The boundary integrals would vanish if $B_0 x^1$ were zero.)

We note that $x^0 = -A$ and since $L_0 A = 0$, $\frac{\partial \bar{p}}{\partial x} = \frac{\partial \bar{p}}{\partial z} = 0$ so that

$$L_1 A = \begin{pmatrix} 0 \\ -\frac{\partial \hat{p}}{\partial x} \\ -\frac{\partial \hat{p}}{\partial z} \\ 0 \end{pmatrix}$$

in both fluids so that

$$\langle C, L_1 x^{(1)} \rangle = -\langle C, L_1 A \rangle$$
$$= \int_{\Omega_1} \bar{u}^* \cdot \nabla \tilde{p} + \int_{\Omega_2} \bar{u}^* \cdot \nabla \tilde{p}$$
$$= \bar{w}^* [\![\tilde{p}]\!], \tag{21}$$

where we denote $C = (\Theta^*, u^*, w^*, h^*)$, and

$$[\![\tilde{p}]\!] = (\bar{M}_1 + P\bar{S}\alpha^2)e^{i\alpha x}. \tag{22}$$

The form of Γ can be read off from the calculation of the adjoint in Appendix A in Ref.1:

$$\Gamma = \int_I \bar{\zeta} e^{i\alpha x} \frac{\partial \bar{\Theta}^*}{\partial z}, \tag{23}$$

where the interval of integration I extends over one wavelength in x, at $z = l_1$, and $\Theta_z^* = \frac{P}{\alpha^2}\Delta^2 w_z^*$. From setting $\Psi = 0$, we have

$$\sigma = -\epsilon \frac{(\Gamma + \bar{w}^* [\![\tilde{p}]\!])}{\langle C, A \rangle} + O(\epsilon^2). \tag{24}$$

We next evaluate $\langle C, A \rangle$:

$$\langle C, A \rangle = \int_I \bar{h}^* e^{i\alpha x} \tag{25}$$

where

$$h^* = -[\![p^*]\!] = -\frac{P}{\alpha^2}[\![w^*_{zzz}]\!]. \tag{26}$$

IV. RESULTS AND DISCUSSION

When $\sinh Q_2 \neq 0$, we have. using the Appendix,

$$\Gamma + \bar{w}^{*}[\![\tilde{p}]\!] = \frac{2\pi}{\alpha}\left(\bar{\varsigma}\frac{P}{\alpha^2}\left[\bar{c}_1 Q_3(Q_3^2 - \alpha^2)^2 \cosh Q_3 l_1 + \bar{c}_3 \bar{Q}_2(Q_2^2 - \alpha^2)^2 \cosh \bar{Q}_2 l_1 + \bar{c}_5 Q_1(Q_1^2 - \alpha^2)^2 \cosh Q_1 l_1\right] + (\bar{M}_1 + \alpha^2 P\bar{S})\left[\bar{c}_1 \sinh Q_3 l_1 + \bar{c}_3 \sinh \bar{Q}_2 l_1 + \bar{c}_5 \sinh Q_1 l_1\right]\right) \tag{27}$$

and

$$[\![w^{*}_{0zzz}]\!] = Q_1^3(c_1 \cosh Q_1 l_1 - d_1 \cosh Q_1 l_2) + Q_2^3(c_3 \cosh Q_2 l_1 - d_3 \cosh Q_2 l_2) + Q_3^3(c_5 \cosh Q_3 l_1 - d_5 \cosh Q_3 l_2), \tag{28}$$

which, on using equations (A15), becomes

$$= d_5 Q_3 \frac{3}{2}(R\alpha^2)^{1/3}(-1 + i\sqrt{3})\frac{\sinh Q_3}{\sinh Q_3 l_1}. \tag{29}$$

Hence,

$$\langle C, A\rangle = \frac{2\pi}{\alpha}\frac{P}{\alpha^2}\bar{d}_5 Q_1 \frac{3}{2}(R\alpha^2)^{1/3}(1 + i\sqrt{3})\frac{\sinh Q_1}{\sinh Q_1 l_1}, \tag{30}$$

and

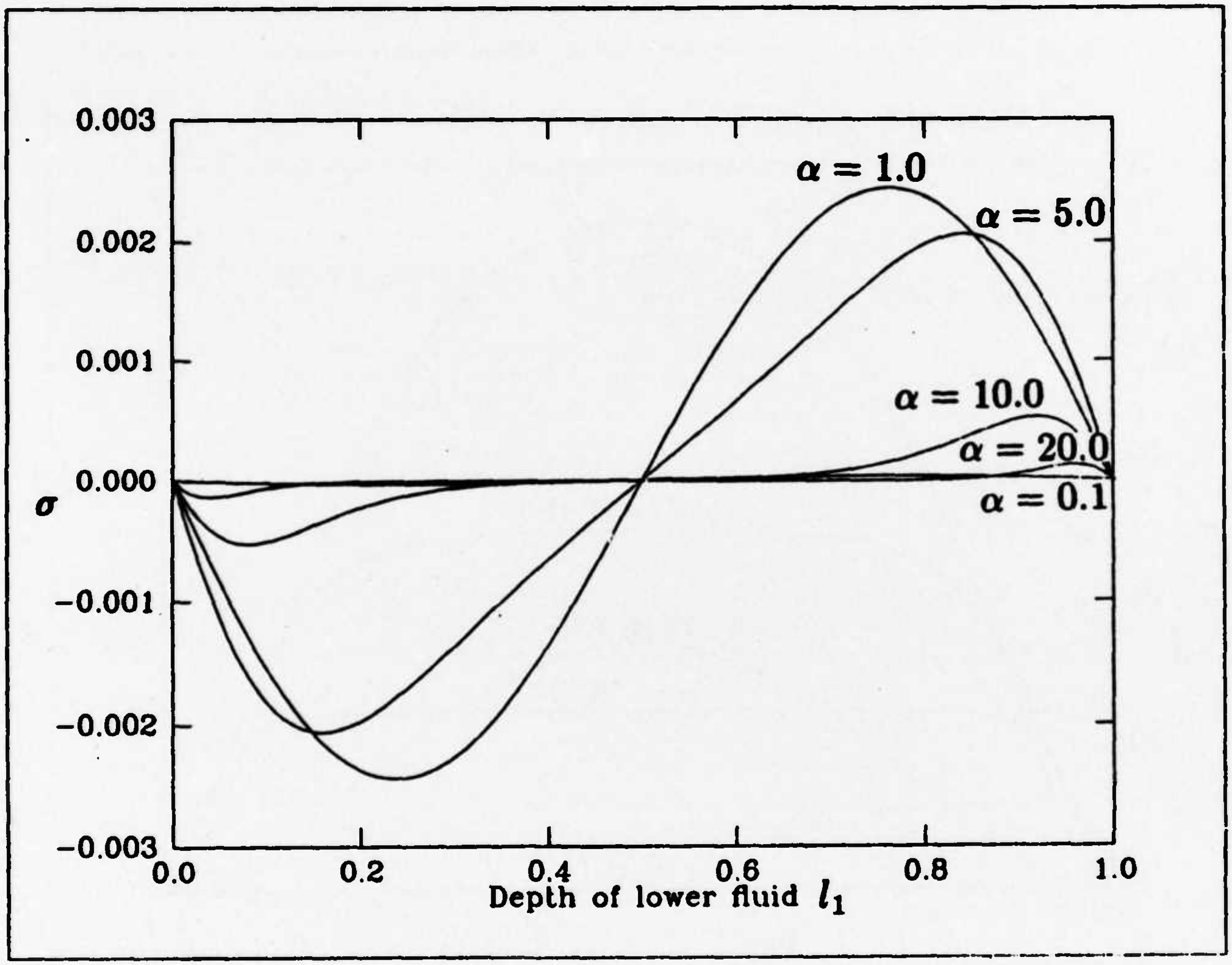
$$\sigma \sim -\frac{\epsilon}{6}\left(\bar{\varsigma}(R\alpha^2)^{1/3}\left[-(1 - i\sqrt{3})\frac{\sinh Q_3 l_2 \cosh Q_3 l_1}{\sinh Q_3} + 2\frac{\sinh \bar{Q}_2 l_2 \cosh \bar{Q}_2 l_1}{\sinh \bar{Q}_2} - (1 + i\sqrt{3})\frac{\sinh Q_1 l_2 \cosh Q_1 l_1}{\sinh Q_1}\right] + \frac{\alpha^2}{(R\alpha^2)^{1/3}}(\bar{M}_1/P + \alpha^2\bar{S})\left[-(1 + i\sqrt{3})\frac{\sinh Q_3 l_2 \sinh Q_3 l_1}{Q_3 \sinh Q_3} + 2\frac{\sinh \bar{Q}_2 l_2 \sinh \bar{Q}_2 l_1}{\bar{Q}_2 \sinh \bar{Q}_2} - (1 - i\sqrt{3})\frac{\sinh Q_1 l_2 \sinh Q_1 l_1}{Q_1 \sinh Q_1}\right]\right) + O(\epsilon^2) \quad \text{as} \quad \epsilon \to 0. \tag{31}$$

We note that σ is real. It is independent of the Prandtl number of the basic one-fluid flow, the perturbation on the viscosity, and the perturbation on thermal diffusivity. The coefficient of $\bar{\varsigma}$ in σ is antisymmetric about $l_1 = 0.5$ (see Figure 1); the coefficients of $\tilde{M}_1/P$ and $\bar{S}$ are symmetric about $l_1 = 0.5$ (see Figures 2-4). In $\tilde{M}_1/P$, $\hat{\alpha}_1 \Delta T^*$ is small compared with 1. In numerical computations, we have set $\hat{\alpha}_1 \Delta T^* = 0.001$. Hence, the coefficient of $\bar{r}$ in σ will be almost symmetric about $l_1 = 0.5$ (see Figure 2). The coefficient of $\bar{\beta}$ in σ is the product of l_2 and a term symmetric about $l_1 = 0.5$ (see Figure 3).

If $Q_2 = 0$, then the two terms in (27) involving $\bar{c}_3$ are replaced by $\alpha^4 \bar{c}_3$ and $l_1 \bar{c}_3$ respectively, and the term involving Q_2 in (28) vanishes. Equation (30) remains as is. In (31), the two terms containing $\bar{Q}_2$ in the square brackets are to be replaced by $2l_2$ and $2l_2 l_1$ respectively. The symmetries discussed in the preceding paragraph still hold.

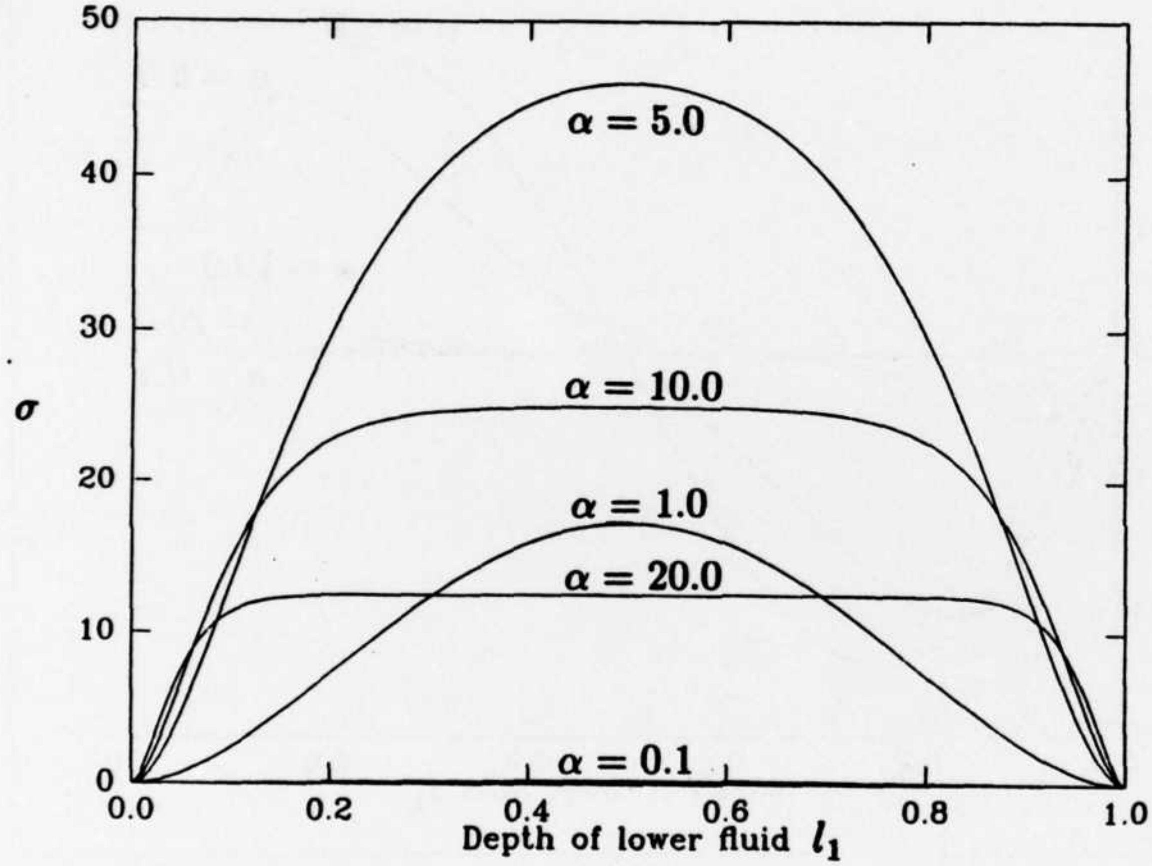
If $Q_2^2 = -j^2\pi^2$, j =non-zero integer, i.e., $\sinh Q_2 = 0$, then as pointed out after equation (A8), c_3 and d_3 are infinite. In $\langle C, A \rangle$, d_3 is multiplied by $\sinh Q_2$ so $\langle C, A \rangle$ is bounded when the basic one-fluid problem is at neutral stability, making the numerator $\Gamma + \tilde{w}^* [\![\tilde{p}]\!]$ unbounded due to the $\bar{c}_3$-term. In this case, our perturbation procedure is not valid, and must be replaced by the perturbation procedure for a double zero eigenvalue[1].

Figure 1



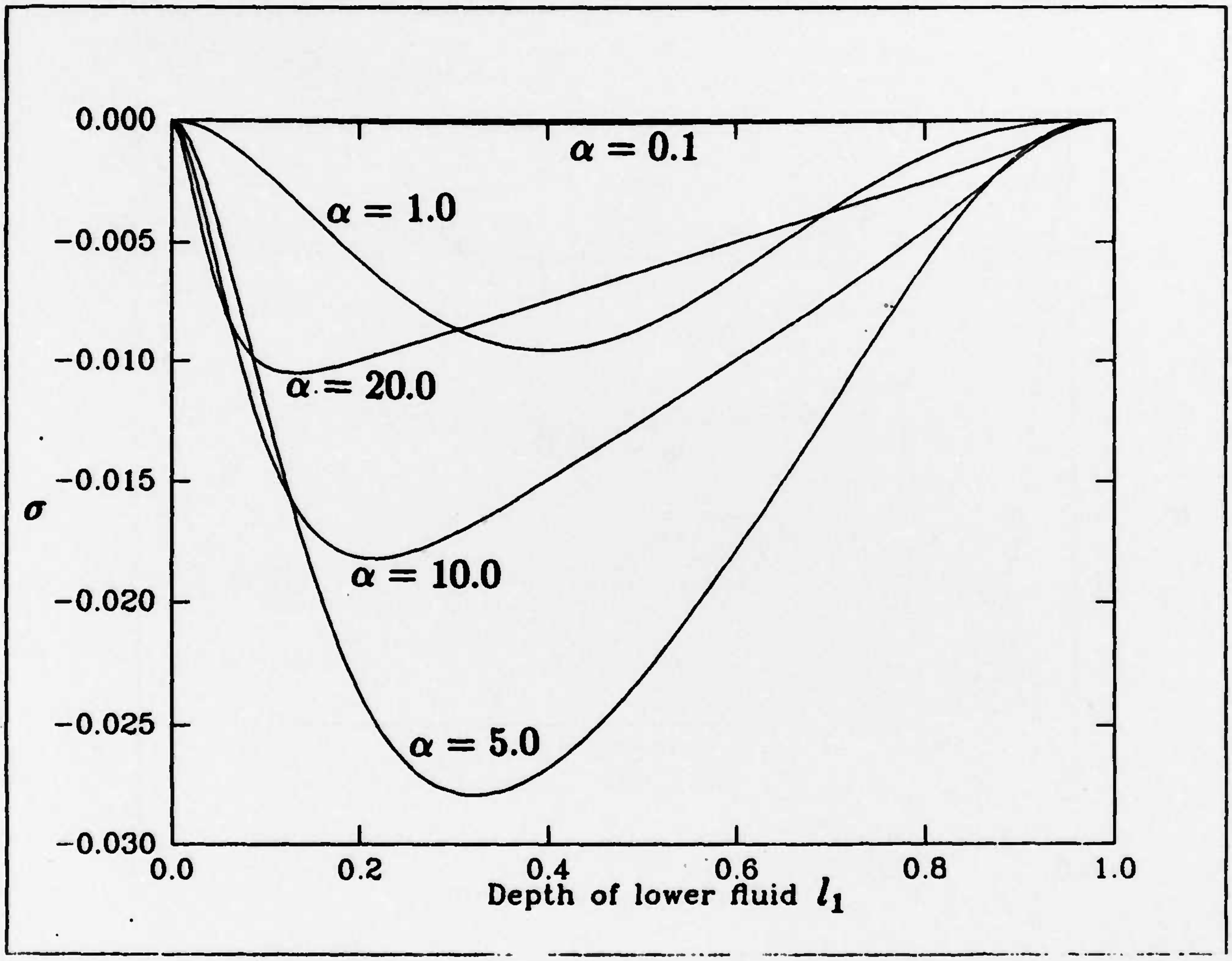
Graph of σ versus l_1, the depth of the lower fluid. $\alpha_1 \Delta T^* = 0.001$, $R = 1$, $\hat{\varsigma} = 1$, $\alpha = 0.1, 1.0, 5.0, 10.0, 20.0$.

Figure 2



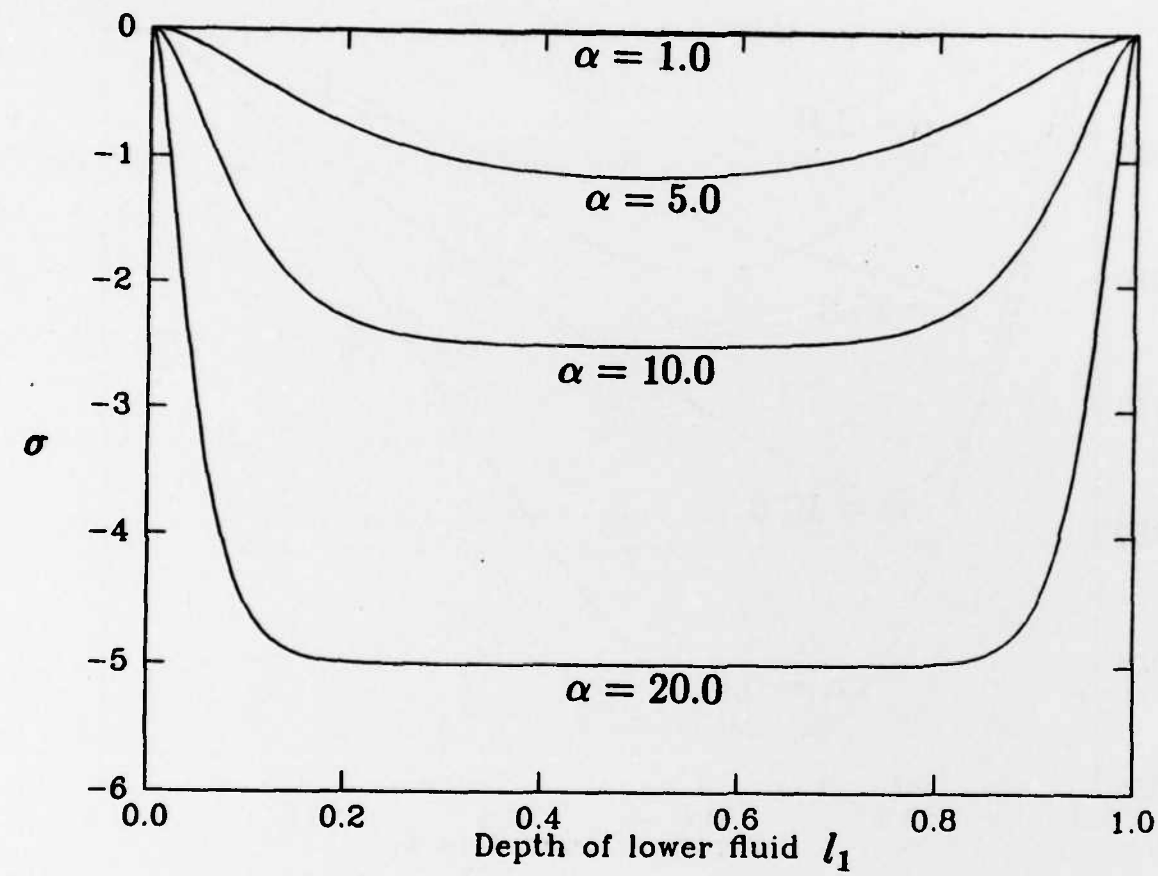
Graph of σ versus l_1, the depth of the lower fluid. $\alpha_1 \Delta T^{\cdot} = 0.001$. $R = 1$, $\bar{r} = 1$, $\alpha = 0.1, 1.0, 5.0, 10.0, 20.0$.

Figure 3



Graph of σ versus l_1, the depth of the lower fluid. $\alpha_1 \Delta T^{*} = 0.001$. $R = 1$, $\bar{\beta} = 1$, $\alpha =$ $0.1, 1.0, 5.0, 10.0, 20.0$.

Figure 4



Graph of σ versus l_1, the depth of the lower fluid. $\alpha_1 \Delta T^{\cdot} = 0.001$, $R = 1$, $\bar{S} = 1$, $\alpha = 0.1, 1.0, 5.0, 10.0, 20.0$.

Short wave asymptotics

When α is large, the boundary conditions at $z = 0, 1$ are irrelevant and the asymptotic behavior of σ is the same as in the case of solid boundaries. The following expansion was derived (Eq.(29), Ref. 2) by scaling the z-variable with $1/\alpha$ and taking the distinguished limit $\alpha \to \infty$, $\bar{S}\alpha^2 = O(1)$:

$$\sigma \sim \frac{R}{2\alpha(\frac{1}{m}+1)}\left(\frac{(\frac{1}{r}-1)}{\hat{\alpha}_1 \Delta T^{\cdot}} + l_2 A_2(1-\frac{1}{r\beta}) - \frac{\alpha^2 S}{R}\right) + O(\frac{1}{\alpha^2})$$

$$\sim \frac{R\epsilon}{4\alpha}\left(\frac{\bar{r}}{\hat{\alpha}_1 \Delta T^{\cdot}} - l_2(\bar{r}+\bar{\beta}) - \frac{\alpha^2 \bar{S}}{R}\right) + O(\frac{1}{\alpha^2}, \epsilon^2). \qquad (32)$$

If $\bar{S}\alpha^2$ is larger than O(1), then it will be the dominant term in the asymptotic expansion but the other terms in (32) will not necessarily be the correct next-order terms.

The numerical calculations checked with this formula for l_1 sufficiently far away from 0 and 1, i.e., for waves that are short enough so that they do not to feel the presence of the boundaries. The coefficient of $-\frac{\epsilon}{6}\bar{\varsigma}(R\alpha^2)^{1/3}$ in equation (31) behaves as

$$\left(-(1-i\sqrt{3})\frac{1}{2}(e^{-2l_1 Q_3} - e^{-2l_2 Q_3}) + e^{-2l_1 \bar{Q}_2} - e^{-2l_2 \bar{Q}_2}\right.$$

$$\left.-(1+i\sqrt{3})\frac{1}{2}(e^{-2l_1 Q_1} - e^{-2l_2 Q_1})\right)\left(1 + O(e^{-2\alpha})\right) \quad \text{as} \quad \alpha \to \infty \qquad (33)$$

so is exponentially small in $|\alpha|$.

Long wave asymptotics

For long wave disturbances, the effect of surface tension is $O(\alpha^4)$ and, provided $Q_2 \neq 0$,

$$\sigma \sim \epsilon R\alpha^2 l_2\left(-\frac{\bar{\varsigma}}{360}(15l_1^4 + 30l_1^2(l_2^2-1) + 7 + l_2^2(-10+3l_2^2))\right.$$

$$\left.+\frac{l_1}{6}(-\frac{\bar{r}}{\hat{\alpha}_1 \Delta T^{\cdot}} + l_2(\bar{r}+\bar{\beta}))(l_1^2 + l_2^2 - 1)\right)$$

$$+O(\alpha^4) \quad \text{as} \quad \alpha \to 0. \qquad (34)$$

When $Q_2 = 0$, the above dominant terms are $O(\alpha^6)$ and

$$\sigma \sim \epsilon\alpha^4 \frac{l_2 l_1}{6}(l_1^2 - l_2^2 - 1)\bar{S} + O(\alpha^6) \qquad \text{as } \alpha \to 0. \tag{35}$$

Thin-layer effects

In the linear stability analysis of parallel shear flows composed of two layers of fluids with different viscosities and densities, the effect of gravity on the density stratification can be countered by the viscosity stratification[5]. A linearly stable arrangement is possible with the more dense fluid being the lower fluid if it is also very much less viscous than the upper fluid and if that layer is sufficiently thin. In the present problem, the basic flow has no shear, but we find that the stability of thin layers can also be counter to intuition. Here, the role of viscosity stratification is taken over by the thermal conductivity stratification. The value of σ for l_1 small is essentially $l_1 \frac{\partial \sigma}{\partial l_1}(l_1 = 0)$ where

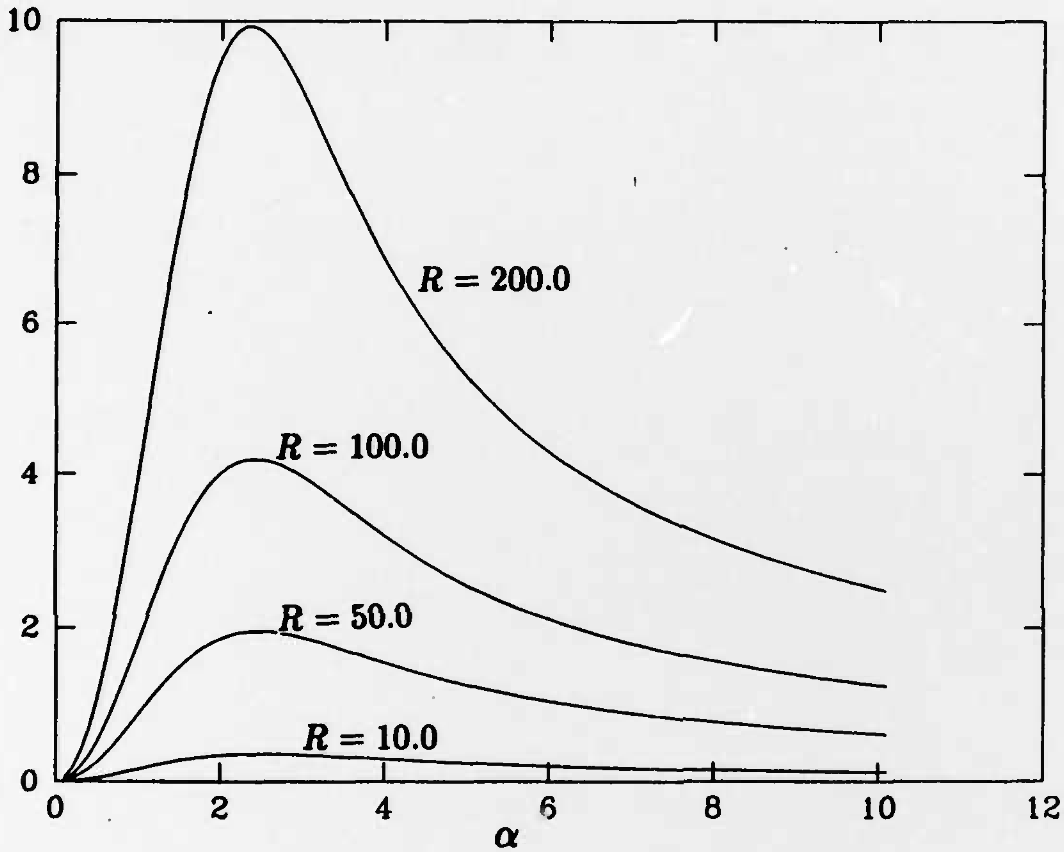
$$\frac{\partial \sigma}{\partial l_1}(l_1 = 0) = -\frac{\epsilon}{3}\bar{\varsigma}(R\alpha^2)^{1/3}\left(Real[(1 + i\sqrt{3})Q_1 \coth Q_1] - \bar{Q}_2 \coth \bar{Q}_2 \right) + O(\epsilon^2). \tag{36}$$

Hence, in the presence of a thin layer, the effect of surface tension and the differences in density and cubical expansions are dominated by the difference in thermal conductivity. The coefficient of $-\epsilon\bar{\varsigma}$ in (36) is positive for Rayleigh numbers up to the first critical value $27\pi^4/4$ for the one-fluid problem, and typically looks like Figure 5. At the first critical value of the Rayleigh number, the coefficient has a pole and is infinite at the critical value of $\alpha = \pi/\sqrt{2}$. As the Rayleigh number increases above the first criticality, there will be two values of α for which the one-fluid problem is neutrally stable[3], and hence the coefficient will have two poles, until the Rayleigh number reaches that of the second criticality, when the coefficient will have three poles, and so on. Our perturbation scheme is valid away from the poles.

For small l_1, $\sigma < 0$, and by antisymmetry, for small l_2, $\sigma > 0$. Hence, the arrangement with a thin layer of a less dense fluid lying below the more dense fluid is stable to long and order 1 wavelength disturbances, if the lower fluid has the lesser thermal conductivity.

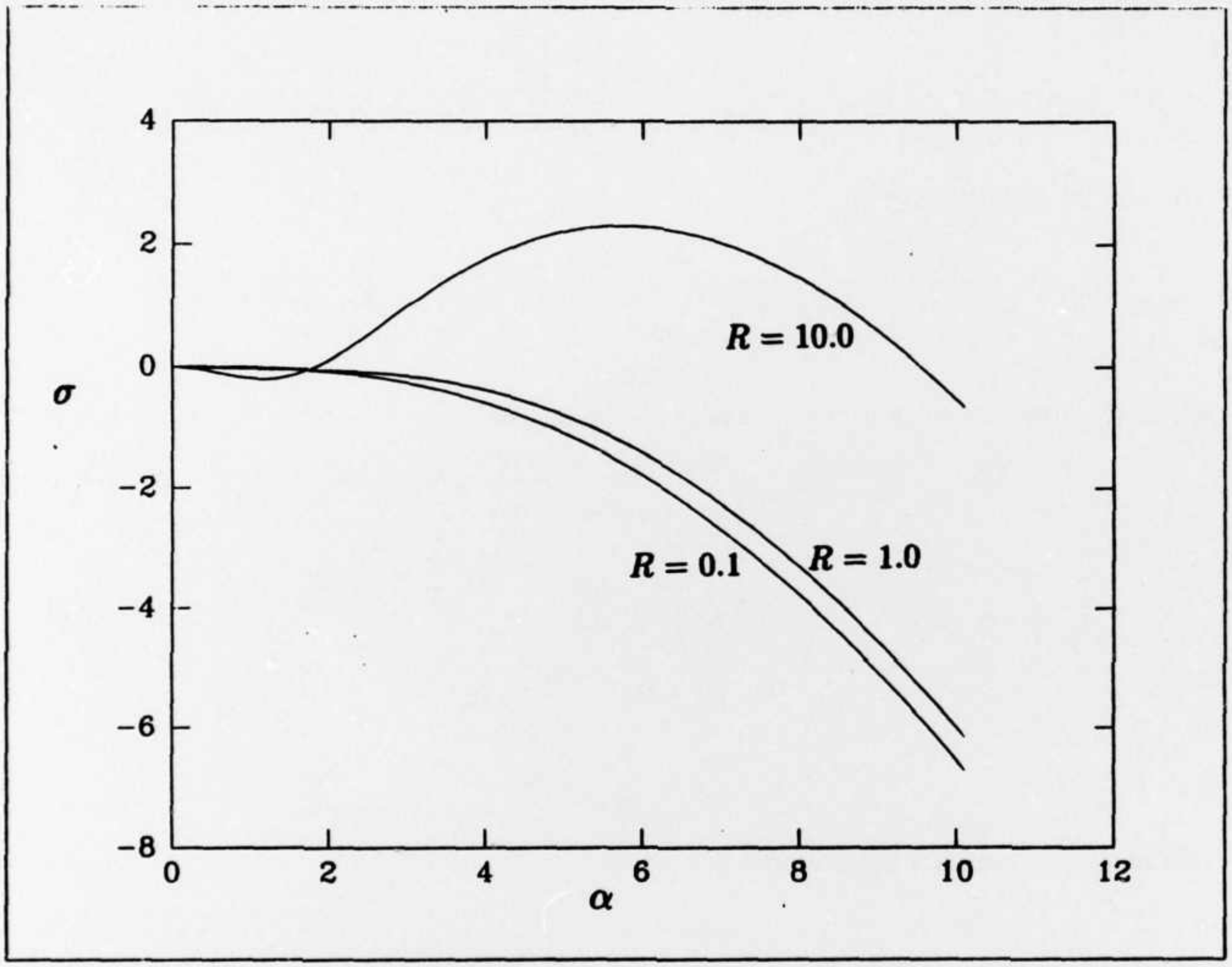
However, for short wave disturbances, the effect of the thermal conductivities is exponentially small, and the density difference and surface tension dominate the stability criterion. Figure 6 displays $\sigma(\alpha)$ for $\bar{r} = 0.1$, $\bar{\varsigma} = 100.0$, $\bar{\beta} = 1.0$, $\bar{S} = 10.0$, $R = 0.1$ and $l_1 = 0.05$ showing linear stability even though Fluid 1 is the less dense.

Figure 5



Graph of the coefficient of $-\epsilon\bar{\varsigma}$ in $\frac{\partial\sigma}{\partial l_1}(l_1 = 0)$ versus α for $R = 10.0, 50.0, 100.0, 200.0$. ıe amplitudes decay to zero for large α. As R approaches $27\pi^4/4$, the peak amplitude proaches infinity and the location of the peak approaches $\alpha = \pi/\sqrt{2}$.

Figure 6



Graph of σ versus α. $\bar{r} = 0.1$, $\bar{\zeta} = 100.0$, $\bar{\beta} = 1.0$, $\bar{S} = 10.0$, $l_1 = 0.05$, $R = 0.1, 1.0, 10.0$. Fluid 1 is the less dense fluid. At $R = 0.1$ and 1.0, the arrangement is linearly stable for all α. As R increases, the arrangement will become less stable. For example, at $R = 10.0$, the arrangement is unstable.

APPENDIX : Eigenfunctions of the unperturbed problem

If $\epsilon=0$, the variable h does not occur in the right hand sides of (4), (5) ,(8), or in the interface conditions, and we have the eigenfunction

$$A = e^{i\alpha x} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{A1}$$

The adjoint equations yield:

$$L_0^* C = 0,$$

$$B_0^* C = 0,$$

where $C = (\Theta^*, u^*, w^*, h^*)$. This leads to the equations[1]:

$$\begin{aligned} \triangle\Theta^* + RPw^* &= 0, \\ P\triangle u^* - \frac{\partial p^*}{\partial x} &= 0, \\ P\triangle w^* + \Theta^* - \frac{\partial p^*}{\partial z} &= 0, \\ \frac{\partial u^*}{\partial x} + \frac{\partial w^*}{\partial z} &= 0. \end{aligned} \tag{A2}$$

Boundary and interface conditions are[1]:

$$\Theta^* = w^* = \frac{\partial u^*}{\partial z} = 0 \qquad \text{at} \qquad z = 0, 1, \tag{A3}$$

$$\text{at} \quad z = l_1, \qquad [\![\Theta^*]\!] = 0. \tag{A4}$$

$$[\![\frac{\partial \Theta^*}{\partial z}]\!] = 0.$$

$$[\![u^*]\!] = 0.$$

$$[\![w^*]\!] = 0.$$

$$[\![\frac{\partial u^*}{\partial z} + \frac{\partial w^*}{\partial x}]\!] = 0,$$

$$[\![p^{\cdot} - 2P\frac{\partial w^{\cdot}}{\partial z}]\!] + h^{\cdot} = 0.$$

We set $w^{\cdot} = w_0^{\cdot} e^{i\alpha x}$ etc., and obtain by combining the equations:

$$(\frac{d^2}{dz^2} - \alpha^2)^3 w_0^{\cdot} + R\alpha^2 w_0^{\cdot} = 0. \tag{A5}$$

The general solution of this equation is

$$w_0^{\cdot} = c_1 \sinh Q_1 z + c_2 \cosh Q_1 z + c_3 \sinh Q_2 z + c_4 \cosh Q_2 z$$

$$+ c_5 \sinh Q_3 z + c_6 \cosh Q_3 z \tag{A6}$$

in fluid 1, and

$$w_0^{\cdot} = d_1 \sinh Q_1(z-1) + d_2 \cosh Q_1(z-1) + d_3 \sinh Q_2(z-1) + d_4 \cosh Q_2(z-1)$$

$$+ d_5 \sinh Q_3(z-1) + d_6 \cosh Q_3(z-1) \tag{A7}$$

in fluid 2, where Q_1 is the complex conjugate of Q_3,

$$Q_1^2 = \alpha^2 + (R\alpha^2)^{1/3} e^{i\pi/3},$$

$$Q_2^2 = \alpha^2 - (R\alpha^2)^{1/3},$$

$$Q_3^2 = \alpha^2 + (R\alpha^2)^{1/3} e^{-i\pi/3}. \tag{A8}$$

The case when $Q_2 = 0$ is considered later (see equations A16-A18). When $Q_2^2 = -\pi^2$, $\sinh Q_2 = 0$, and the problem reduces to that considered in Ref. 1, i.e. the perturbation of a double zero eigenvalue. This occurs first at $R = \frac{27\pi^4}{4}$ and $\alpha = \pi/\sqrt{2}$, and the Q_1 and Q_3 above reduce to Q_1 and Q_2 of equation (B8) of Ref. 1. In fact, $\sinh Q_2 = 0$ for $R = (j^2\pi^2 + \alpha^2)^3/\alpha^2$, j =non-zero integer, so that when the basic one-fluid problem is at neutral stability, our c_3 and d_3 are ∞.

The coefficients c_1 - c_6 and d_1 - d_6 must be determined such that the boundary conditions are satisfied. The conditions (A3) at the walls reduce to

$$w_0^{\cdot} = \frac{d^2}{dz^2} w_0^{\cdot} = \frac{d^4}{dz^4} w_0^{\cdot} = 0. \tag{A9}$$

At z=0, this yields

$$c_2 + c_4 + c_6 = 0,$$

$$Q_1^2 c_2 + Q_2^2 c_4 + Q_3^2 c_6 = 0, \qquad (A10)$$

$$Q_1^4 c_2 + Q_2^4 c_4 + Q_3^4 c_6 = 0.$$

From thi e obtain $c_2 = c_4 = c_6 = 0$. At z=1, we find

$$d_2 + d_4 + d_6 = 0,$$

$$Q_1^2 d_2 + Q_2^2 d_4 + Q_3^2 d_6 = 0, \qquad (A11)$$

$$Q_1^4 d_2 + Q_2^4 d_4 + Q_3^4 d_6 = 0.$$

This yields $d_2 = d_4 = d_6 = 0$. The first five of conditions (A4) lead, after eliminating u^* and Θ^*, to the conditions

$$[\![w_0^*]\!] = [\![\frac{dw_0^*}{dz}]\!] = [\![\frac{d^2 w_0^*}{dz^2}]\!] = [\![\frac{d^4 w_0^*}{dz^4}]\!]$$

$$= [\![\frac{d^5 w_0^*}{dz^5} - 2\alpha^2 \frac{d^3 w_0^*}{dz^3}]\!] = 0. \qquad (A12)$$

We thus obtain the following system of equations.

$$c_1 \sinh Q_1 l_1 + c_3 \sinh Q_2 l_1 + c_5 \sinh Q_3 l_1 = -d_1 \sinh Q_1 l_2 - d_3 \sinh Q_2 l_2 - d_5 \sinh Q_3 l_2,$$

$$c_1 Q_1 \cosh Q_1 l_1 + c_3 Q_2 \cosh Q_2 l_1 + c_5 Q_3 \cosh Q_3 l_1 = Q_1 d_1 \cosh Q_1 l_2$$

$$+ Q_2 d_3 \cosh Q_2 l_2 + Q_3 d_5 \cosh Q_3 l_2,$$

$$c_1 Q_1^2 \sinh Q_1 l_1 + c_3 Q_2^2 \sinh Q_2 l_1 + c_5 Q_3^2 \sinh Q_3 l_1 = -Q_1^2 d_1 \sinh Q_1 l_2$$

$$-Q_2^2 d_3 \sinh Q_2 l_2 - Q_3^2 d_5 \sinh Q_3 l_2,$$

$$c_1 Q_1^4 \sinh Q_1 l_1 + c_3 Q_2^4 \sinh Q_2 l_1 + c_5 Q_3^4 \sinh Q_3 l_1 = -Q_1^4 d_1 \sinh Q_1 l_2$$

$$-Q_2^4 d_3 \sinh Q_2 l_2 - Q_3^4 d_5 \sinh Q_3 l_2. \qquad (A13)$$

$$c_1(Q_1^5 - 2\alpha^2 Q_1^3)\cosh Q_1 l_1 + c_3(Q_2^5 - 2\alpha^2 Q_2^3)\cosh Q_2 l_1 + c_5(Q_3^5 - 2\alpha^2 Q_3^3)\cosh Q_3 l_1$$

$$= d_1(Q_1^5 - 2\alpha^2 Q_1^3)\cosh Q_1 l_2 + d_3(Q_2^5 - 2\alpha^2 Q_2^3)\cosh Q_2 l_2 + d_5(Q_3^5 - 2\alpha^2 Q_3^3)\cosh Q_3 l_2.$$

From these, we find the following relations that will be useful later:

$$\begin{aligned} c_1 &= -d_1 \frac{\sinh Q_1 l_2}{\sinh Q_1 l_1}, \\ c_3 &= -d_3 \frac{\sinh Q_2 l_2}{\sinh Q_2 l_1}, \\ c_5 &= -d_5 \frac{\sinh Q_3 l_2}{\sinh Q_3 l_1}. \end{aligned} \tag{A14}$$

We express the coefficients in terms of d_5:

$$\begin{aligned} c_1 &= d_5 \frac{Q_3 \sinh Q_3 \sinh Q_1 l_2}{Q_1 \sinh Q_1 \sinh Q_3 l_1} \frac{(1 + i\sqrt{3})}{2} \\ c_3 &= d_5 \frac{Q_3 \sinh Q_3 \sinh Q_2 l_2}{Q_2 \sinh Q_2 \sinh Q_3 l_1} \frac{(1 - i\sqrt{3})}{2} \\ c_5 &= -d_5 \frac{\sinh Q_3 l_2}{\sinh Q_3 l_1} \\ d_1 &= -d_5 \frac{Q_3 \sinh Q_3 \sinh Q_1 l_1}{Q_1 \sinh Q_1 \sinh Q_3 l_1} \frac{(1 + i\sqrt{3})}{2} \\ d_3 &= -d_5 \frac{Q_3 \sinh Q_3 \sinh Q_2 l_1}{Q_2 \sinh Q_2 \sinh Q_3 l_1} \frac{(1 - i\sqrt{3})}{2}. \end{aligned} \tag{A15}$$

When $\alpha^2 = (R\alpha^2)^{1/3}$, $Q_2 = 0$. The general solution of (A5) is

$$w_0' = c_1 \sinh Q_1 z + c_3 z + c_5 \sinh Q_3 z \qquad \text{in fluid 1}$$

and

$$w_0' = d_1 \sinh Q_1(z-1) + d_3(z-1) + d_5 \sinh Q_3(z-1) \qquad \text{in fluid 2.} \tag{A16}$$

As expected, from taking the limit as $Q_2 \to 0$ in (A14),

$$c_3 = -\frac{l_2}{l_1} d_3. \tag{A17}$$

Hence, (A15) holds for c_1, c_5 and d_1, and

$$c_3 = d_5 \frac{Q_3 l_2 \sinh Q_3}{\sinh Q_3 l_1} \frac{(1 - i\sqrt{3})}{2}. \tag{A18}$$

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A linear stability analysis of the Bénard problem for two layers of different fluids lying on top of each other and bounded by free surfaces is considered. The fluids are assumed to be similar and perturbation methods are used to calculate the eigenvalue in closed form. The case of the Rayleigh number and wavenumber of the disturbance being close to the first criticality (cont.)

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ABSTRACT (cont.)

of the one-fluid Bénard problem has been investigated in a previous paper[1], and was found to exhibit both overstability and convective instability. In this paper, the Rayleigh number is assumed to be less than that of the first criticality of the one-fluid problem, and in this situation, overstability does not occur. An unexpected result is that by an appropriate choice of parameters, it is possible to find linearly stable arrangements with the more dense fluid on top.

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